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Does stochastic resonance occur in periodic potentials?

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The possibility of observing stochastic resonance (SR) in a periodic potential without a static bias, driven by noise and an oscillating field, is investigated in both limits of damping. It is shown from the matrix continued fraction method that, although an underdamped Brownian particle moving in a periodic potential displays a dynamical resonance facilitated by noise, there is no conventional SR irrespective of damping. The reason such SR cannot be observed in an overdamped regime is demonstrated by a hopping model. Due to the unbound motion in the periodic potential, transition probability decays algebraically and there is no persistent synchronized hopping. However, the noise-induced enhancement of the diffusion constant related to escape rate enhancement exhibits an SR-like behavior. A comparison with the dynamics on a circle as a different example of an unbound system is given. [S1063-651X(98)50706-4]

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The stochastic resonance (SR) is a synchronization of a dynamical variable, for example, the position of a Brownian particle, with external driving force, which emerges as a cooperation of the external driving and random noise in nonlinear dynamical systems. Contrary to common sense, the coherent response of a system to an external driving force can be enhanced by the noise, tending to be the maximum at an optimal noise strength, in which the period of external driving matches the internal time scale.

Since SR was originally proposed to explain the periodic recurrences of the earth's ice ages [1], there have been many theoretical developments of SR in conventional bistable systems. McNamara and Wiesenfeld [2] have suggested a master equation for the populations in two stable states. While they considered the signal-to-noise ratio, i.e., the ratio of the δ peak height in the power spectrum to the noise background as a probe of SR effect, Zhou, Moss, and Jung [3] have suggested the escape time distribution to describe SR. Jung and Hänggi [4] described SR within the framework of nonstationary stochastic processes without restriction to small driving amplitudes or frequencies, where they presented power spectral densities and signal amplification as measures of SR. They also have predicted the subharmonics in the power spectrum and its selection rules for symmetric potentials.

There are a great deal of applications of SR, ranging from a global climate model [1], electronic circuits, e.g., Schmitt triggers [5], and a bidirectional ring laser [6] to biological systems [7,8]. While the overwhelming majority of the work has been concerned with bistable systems, SR is found to be more widespread in nature. For example, nonconventional SR has been reported in monostable wells where inertial effects are important [9]. Wiesenfeld *et al.* [10] demonstrated that SR is expected to be observable in excitable dynamics with deterministic reinjection, ''threshold-plus-reinjection dynamics,'' involving the dynamics on a circle that is multistable. However, the presence of SR in periodic potential is still questionable.

The dynamics of a Brownian particle in a periodic potential is an important problem with many applications in physics and chemistry [11], such as motion of ions in superionic conductors [12], fluctuations of the Josephson supercurrent in tunneling junctions [13], and relaxation dynamics of rotators [14]. Although SR and the associated noise-induced transport in a periodic potential with a bias have been investigated recently [15,16], the SR driven only by time periodic external force is not clearly understood yet. The main purpose of this work is to investigate the possibility of SR for the dynamics in periodic potential for various ranges of damping.

The one-dimensional Brownian motion of a tagged particle of mass m in a potential $U(x)$ under external driving $A\cos\omega t$ can be described by the Langevin equation

$$m\ddot{x}+m\gamma\dot{x}+\frac{\partial U(x)}{\partial x}=A\cos\omega t+R(t), \tag{1}$$

where γ is the damping constant, the random force $R(t)$ is a Gaussian white noise connected with a noise strength D via the fluctuation-dissipation theorem

$$\langle R(t)R(0)\rangle=2m\gamma k_BT\delta(t)=2D\delta(t). \tag{2}$$

As an example of the periodic potential with periodicity a, $U(x+a)=U(x)$, we consider a sinusoidal potential

$$U(x)=-\frac{1}{2}\Delta U\cos\frac{2\pi}{a}x=-m\omega_0^2\left(\frac{a}{2\pi}\right)^2\cos\frac{2\pi}{a}x, \tag{3}$$

where $\omega_0=2\pi/a\sqrt{\Delta U/2m}$ is the natural frequency at the bottom of the potential well and ΔU is the potential barrier height. Both $-[\partial U(x)]/\partial x$ and $R(t)$ originate from the bath and describe respectively the systematic and rapidly fluctuating parts of the force on the tagged particle. The Fokker-Planck equation that is equivalent to the Langevin equation [Eq. (1)] is

$$\frac{\partial P(x,v,t)}{\partial t}=\mathcal{L}_{FP}P(x,v,t), \tag{4}$$

where

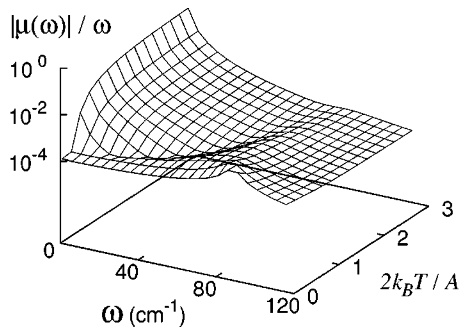

FIG. 1. Signal power amplification factor as a function of frequency and noise strength in underdamped regime $\gamma=5\ \mathrm{cm}^{-1}$. The characteristic frequency of potential ω_0 is 88 cm^{-1}, which remains the same throughout.

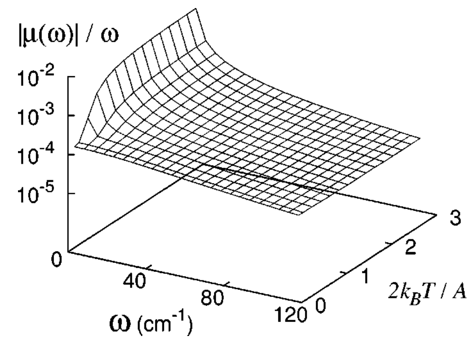

FIG. 2. Signal power amplification factor in an overdamped regime $\gamma=200\ \mathrm{cm}^{-1}$. The power amplification increases monotonically with respect to the noise strength around the interwell hopping frequency regime $\mathcal{O}(1)\ \mathrm{cm}^{-1}$.

$$\mathcal{L}_{FP}=-v\frac{\partial}{\partial x}+\left(\frac{\partial U(x)}{m\partial x}-A\cos\omega t\right)\frac{\partial}{\partial v}+\frac{\partial}{\partial v}\left(\gamma v+\gamma\frac{k_BT}{m}\frac{\partial}{\partial v}\right). \tag{5}$$

Now suppose that the external driving force has a small amplitude so that linear-response theory (LRT) is applied. By LRT, the response of the system to the perturbing influence of an external field can be described entirely in terms of stationary time-correlation functions that are characteristic of the system in the absence of the field. LRT makes it possible to predict the onset of SR solely in terms of the spectral density of the correlation in the absence of periodic driving and its dependence on temperature (noise strength). The signal power amplification $\eta(\omega)$, a measure of SR, is proportional to the absolute square of $\chi(\omega)$ [17], a dynamic response function (frequency-dependent susceptibility) of position to external force, which is related to the Fourier transform of velocity correlation function (frequency-dependent mobility), $\mu(\omega)=-i\omega\chi(\omega)$, via

$$\eta(\omega)=|\chi(\omega)|^2=\frac{1}{\omega^2}|\mu(\omega)|^2. \tag{6}$$

In the above,

$$\chi(\omega)=-\frac{1}{k_BT}\int_0^\infty\frac{d}{dt}\langle x(t)x(0)\rangle e^{-i\omega t}dt, \tag{7}$$

$$\mu(\omega)=\frac{1}{k_BT}\int_0^\infty\langle v(t)v(0)\rangle e^{-i\omega t}dt, \tag{8}$$

and $\langle\ \rangle$ denotes the average over the stationary-state ensemble. Due to the relation in Eq. (6), it suffices to consider the time-correlation function of velocity along with the mobility for SR, even if the position has to be the only relevant variable in overdamped systems and the time-correlation functions of position along with the susceptibility have to be considered. Therefore, $|\mu(\omega)|/\omega$ is evaluated as a probe for SR in this paper.

We present the numerical results for the amplitude of frequency-dependent mobility, $|\mu(\omega)|$ from the matrix continued fraction method (MCFM) [11,18]. It is well known that MCFM can provide a nearly exact numerical solution for a Brownian particle in a periodic well [19]. In this study, the number of matrix elements used for numerical evaluation was 23 and it was shown that the convergences of results are good enough to assure their validity as an excellent approximation to the exact one even in the case of very low damping. The $|\mu(\omega)|/\omega$, controlled by the temperature, is shown in a low damping regime, $\gamma=5\ \mathrm{cm}^{-1}$ (Fig. 1), and a high damping regime, $\gamma=200\ \mathrm{cm}^{-1}$ (Fig. 2). We employ the unit in accordance with the convention in superionic conductors and set ω_0 to be 88 cm^{-1}. The $|\mu(\omega)|/\omega$ in a low damping regime has a peak around the natural frequency ω_0, which also shows resonance behavior with respect to the noise strength, i.e., temperature as more explicitly seen in Fig. 3. The resonance behavior is more pronounced as damping decreases. However, this resonance behavior is not the SR associated with interwell hopping, but a noise-facilitated standard dynamical resonance, mentioned by Dykman *et al.* [9], which originates from the intrawell motion. The interwell hopping frequency is much smaller than ω_0 and is estimated to be $\mathcal{O}(1)\ \mathrm{cm}^{-1}$ according to the escape rate in bimetastable potential [20]. Since the power amplification increases monotonically as a function of temperature around the interwell hopping frequency regime (Fig. 3), it can be concluded that the SR, due to the hopping synchronized with modulation, does not exist in periodic structure.

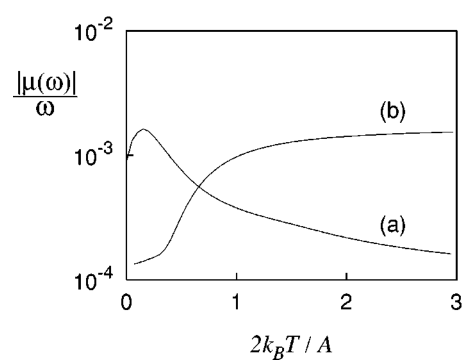

FIG. 3. Curve (a) shows the noise-facilitated standard dynamical resonance for $\omega=83\ \mathrm{cm}^{-1}$ in an underdamped regime $\gamma=5\ \mathrm{cm}^{-1}$. Curve (b) displays the monotonically increasing power amplification for $\omega=3\ \mathrm{cm}^{-1}$ in an overdamped regime $\gamma=200\ \mathrm{cm}^{-1}$.

Below, a hopping model is suggested to rationalize the above conclusion in an overdamped regime. Neglecting the inertia term in Eq. (1), we write the overdamped Langevin equation in dimensionless units as

$$\dot{x} = \cos x + A \cos \omega t + R(t). \tag{9}$$

If the external field is weak and adiabatic, the rate of transition to the next well is assumed to be

$$W(t) \sim e^{-(\Delta U \pm A \cos \omega t)/D} \approx \lambda \pm \varepsilon \cos \omega t, \tag{10}$$

where the zeroth-order transition rate $\lambda = e^{-\Delta U/D}$ and the first-order transition rate $\varepsilon = (A/D)e^{-\Delta U/D}$ are identified. For the probability density of a particle localized at the nth potential well, given by $P_n(t) = \int_{2n\pi-\pi}^{2n\pi+\pi} P(x,t)\, dx$, the master equation reads

$$\begin{aligned}\dot{P}_n &= (\lambda - \varepsilon \cos \omega t)P_{n+1} + (\lambda + \varepsilon \cos \omega t)P_{n-1} - 2\lambda P_n \\ &= \lambda(P_{n+1} + P_{n-1} - 2P_n) - \varepsilon \cos \omega t (P_{n+1} - P_{n-1}).\end{aligned} \tag{11}$$

In solving the equation, the periodic boundary condition $[P_{n+N}(t) = P_n(t)]$ is chosen to assure that the process is stationary. With the particle set at the site $n=0$ initially, the transition probability of finding the particle at the site n after time t is given to the first order of ε straightforwardly,

$$P(n,t|n=0,t_0=0) = e^{-2\lambda t} \sum_{l=-\infty}^{\infty} \left[1 + \frac{\varepsilon}{\omega \lambda t}(n+lN)\sin \omega t \right] I_{n+lN}(2\lambda t), \tag{12}$$

where I_n is the modified Bessel function of order n. The transition probability gives the stationary probability distribution in the absence of the external field $P_{st} = 1/N$ as time goes to infinity. With the periodicity N set to infinity, to recover the extended system of our concern, the transition probability is dominated by the $l=0$ term:

$$\begin{aligned}P(n,t|0,0) &= e^{-2\lambda t}\left[1 + \frac{\varepsilon n}{\omega \lambda t} \sin \omega t \right] I_n(2\lambda t) \quad \text{for large } N, \\ &= \frac{1}{\sqrt{2\pi\lambda t}}\left[1 + \frac{\varepsilon n}{\omega \lambda t} \sin \omega t \right] \quad \text{for large } t, \\ &= \alpha t^{-\frac{1}{2}} + \beta t^{-\frac{3}{2}} \sin \omega t.\end{aligned} \tag{13}$$

As the system size N goes to infinity, the δ peak at driving frequency cannot be detected in the power spectrum. The coherently modulated part of the transition probability in the periodic structure shows the power-law decay $t^{-3/2}$, while the corresponding quantity in the bistable potential has constant amplitude with respect to time, giving rise to the δ peak at driving frequency in the power spectrum [2]. The coherent behavior of the power spectrum in the long-time limit does not occur in this extended system because the coherent part in the transition probability decays algebraically, faster than the noncoherent part. The fact that both parts of the transition probability decay algebraically means that particles diffuse away, which is the most important difference between bound bistable and unbound periodic systems.

The dynamics on a circle modeling action potential events in sensory neurons is known to exhibit SR [10]. It is obviously a multistable system. As only one-directional hopping is, however, allowed, it is a Poisson process. In the presence of the sinusoidal force, Eq. (11), adopted to a one-directional process, yields the Poisson distribution as in Ref. [10],

$$\begin{aligned}P(n,t|0,0) &= e^{-(\lambda t - (\varepsilon/\omega)\sin \omega t)}\left(\lambda t - \frac{\varepsilon}{\omega}\sin \omega t \right)^n \Big/ n! \\ &\cong e^{-\lambda t}(\lambda t)^n \left(1 + \frac{\varepsilon}{\omega}\sin \omega t \right) \Big/ n!.\end{aligned} \tag{14}$$

It is shown that the decay of the coherent part is as slow as the noncoherent part from the noise background, which gives rise to a pronounced peak at the driving frequency in the power spectrum. Though both the periodic and circle systems are unbounded, the difference lies in whether or not ‘‘back and forth’’ hoppings are allowed. When transitions are made only to one side, the dynamics is a kind of threshold phenomenon counting only the number of hopping events or firings during the time interval. In a circle system, the firing event is interpreted as the transmitted information in the same way that the interwell switching event is interpreted as a signal output in bistable potential. However, the position of the Brownian particle, rather than the hopping event itself, has to be considered as signal output in periodic potential because the position after hoppings is not deterministic, unlike the circle system. The answer for SR depends on the characteristics of the system, namely, which dynamical variable is chosen to be considered as signal output.

As mentioned above, what is to be considered to probe SR in the periodic potential is not only the number of events happening, but the position of the particle as a result of hopping events. The coherent component cannot survive in the time-correlation function of positions when particles diffuse away. The effect of the signal is to periodically modulate the hopping rate, which directly affects the diffusion constant. Enhancement of the diffusion constant $\mathcal{D} = \lim_{t\to\infty}\langle [x(t) - x(0)]^2 \rangle / 2t$, when a periodic signal is applied, can easily be obtained in our random walk model as

$$\begin{aligned}\kappa(A,D) &= \frac{\mathcal{D}(A)}{\mathcal{D}(A=0)} \\ &= 1 + \frac{1}{4}\left(1 + \frac{\sin 2\omega t}{2\omega t} \right)\frac{A^2}{D^2} e^{-\Delta U/D},\end{aligned} \tag{15}$$

where $\mathcal{D}(A=0)$ is the diffusion constant in the periodic structure without $A \cos \omega t$. The numerical results of Hu Gang, Daffertshofer, and Haken [21] for diffusion in periodic potential is explained by our analytic expression. The transition rate to the right well is the same as that to the left well to the zeroth order of ε. However, the escape rate (hopping event) itself can be maximized at an optimal noise intensity

$\Delta U \approx D$ only when it is coupled with external driving. Enhancement of the diffusion constant is related to the escape rate enhancement induced by the modulation [17,22] and it displays SR-like behavior.

In conclusion, SR in the conventional sense takes place when the modulation synchronizes hopping and generates a periodic contribution to the jump process. It is shown through the numerical computation, MCFM that there is no such SR in periodic potentials in both limits of damping. The noise-facilitated ordinary dynamical resonance rooted in the motion in the bottom of the well is shown through the power amplification factor in the underdamped limit, as discussed by Dykman *et al.* The reason a coherent hopping contribution cannot be observed in the overdamped limit either is qualitatively argued by the transition rate in our hopping model. It is necessary that the transition probability have a slowly decaying coherent part to the external field in order to generate a periodic contribution in the power spectrum. Since periodic structure is infinitely extended, the transition probability exhibits a power-law decay of the coherently oscillating part. This is different from the cases of bounded systems (bistable wells). In a word, there is no synchronized incessant hopping, namely, SR, because the particle diffuses away. Nevertheless, the enhancement of the diffusion constant shows SR-like behavior due to the fact that the enhancement of the escape rate by the modulation is optimized at a certain noise strength in spite of the absence of directionality.

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